This problem considers asset pricing in a linear production economy much like that of King, Plosser, and Rebelo (1988) and Cox, Ingersoll, and Ross (1985). The pricing model is essentially a discrete time version of Vasicek’s (1977) term-structure model.

Consider the standard neo-classical stochastic growth model. A representative agent solves the following problem.

$$\max_{C_t} E_0 \sum_{t=0}^{\infty} \beta^t u(C_t) ,$$

subject to:

$$Y_t = I_t + C_t$$  \hspace{1cm} (1)

$$Y_{t+1} = f(K_t, v_{t+1})$$  \hspace{1cm} (2)

$$K_t = (1 - \kappa)K_{t-1} + I_t$$  \hspace{1cm} (3)

The notation is as follows. Consumption, output and investment are denoted $C$, $Y$ and $I$, respectively. Output is produced using capital, $K$, according to the production function, $f$, which is subject a stochastic technology shock, $v$. $\beta$ is the discount factor, which equals the inverse of one plus the agent’s rate of time preference. $\kappa$ denotes the rate of depreciation on the capital stock.

Equation (1) is the national income identity. It says that savings must equal investment. Equation (2) is the aggregate production function, while equation (3) is the ‘transition equation’ (or the ‘law of motion’) for the capital stock.

This is the standard model upon which much of modern dynamic macroeconomics is based. We’ll now put the following additional structure on the model.

- The depreciation rate is unity, so that all of the previous period’s capital stock is used up in production. This is a drastic assumption which negates many of the interesting dynamics of the model. It does, on the other hand, allow us to obtain closed form solutions.

- Technology is linear: $f(K_t, v_{t+1}) = K_tv_{t+1}$. 


• Utility is logarithmic: \( u(C) = \log(C) \).

• The technology shocks are conditionally lognormal:\(^1\)

\[
v_{t+1} = \exp(z_t + \epsilon_{t+1}) ,
\]

where \( \epsilon_t \sim N(0, \sigma^2) \) and,

\[
z_t = (1 - \rho)\delta + \rho z_{t-1} + \lambda \epsilon_t .
\]

The variable \( z \) is referred to as the ‘state variable.’ One thing to note at this point is that there is a single source of uncertainty driving the model: the innovations \( \epsilon \). The jargon is that this is a ‘single-factor model.’

(a) Show that the model can now be succinctly written as:

\[
\max_{C_t} E_0 \sum_{t=0}^{\infty} \beta^t u(C_t) ,
\]

subject to:

\[
Y_{t+1} = (Y_t - C_t) \exp(z_t + \epsilon_{t+1})
\]

\[
z_t = (1 - \rho)\delta + \rho z_{t-1} + \lambda \epsilon_t .
\]

(b) Use dynamic programming to solve for the closed-form solution to the model. 

*Hint:* First, guess that the value function has the form,

\[
V(Y, z_t) = F + G \log(Y_t) + Hz_t .
\]

Using this, compute the consumption policy function, \( C(Y_t, z_t) \) (it should depend on \( Y_t \) only). Substitute your answer back into the Bellman equation and *verify* that your guess about the functional form of \( V \) was correct. Having done this, you should be able to express consumption as a stationary function of the model’s parameters and the state variable, \( Y \).

(c) Verify that the pricing kernel for this economy is equal to the inverse of the technology shock. That is,

\[
\pi_{t+1} = \exp(-z_t - \epsilon_{t+1}) .
\]

Show that \( -\log(\pi_t) \) has an ARMA(1,1) structure.

(d) What is the rate of return on capital between periods \( t \) and \( t+1 \)? Note that, this ‘price’ is an equilibrium price, in spite of the fact that it is specified exogenously. This is an artifact of the ‘constant returns’ production technology and is a standard trick in general equilbrium valuation models. This is analogous to specifying an exogenous diffusion equation for returns in, say, the CIR model.

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\(^1\)It will be useful to recall that if \( \log(x) \sim N(\mu, \sigma^2) \) (so that \( x \) is lognormally distributed) then \( E(x) = \exp(\mu + \sigma^2/2) \).
References

