Problem Set 2  
Finance 1, 47-720

1. (beta representation of state pricing model)
   Defining \( m \) as a pricing kernel (as we did in class), we can always find the portfolio, \( \theta^* \) which solves the problem:
   \[
   \sup_{\theta} \text{corr}(D^\top \theta, m)
   \]
   Suppose that the return on the portfolio \( \theta^* \), \( R^* \), has a non-zero variance. Show that, for any other portfolio return, \( R^\theta \),
   \[
   E(R^\theta) - R^I = \beta^\theta (E(R^*) - R^I)
   \]
   where \( R^I = [E(m)]^{-1} \) and \( \beta^\theta \equiv \text{Cov}(R^*, R^\theta)/\text{Var}(R^*) \).
   **Hint:** for any two \( x \) and \( y \) in \( \mathbb{R}^S \) with \( \text{Var}(y) \equiv \text{Cov}(y, y) \neq 0 \) the *linear regression* of \( x \) on \( y \) is uniquely defined by \( x = \alpha + \beta y + \varepsilon \), where \( \beta = \text{Cov}(x, y)/\text{Var}(y) \), and \( \text{Cov}(y, \varepsilon) = E(\varepsilon) = 0 \).

2. (linear separation of cones)
   Define the set \( A \) as follows:
   \[
   A = \{ a \in \mathbb{R}^N : a = Dr, r \in \mathbb{R}^S_{++} \}
   \]
   where \( D \) is the same \( N \times S \) matrix of security payoffs we have been using in class. The prices of the \( N \) securities are denoted \( q \in \mathbb{R}^N \).
   Yet another useful way to think about arbitrage is that the set \( A \) can be separated from \( q \). That is, there is a vector, \( b \in \mathbb{R}^N \), such that \( q \cdot b \leq 0 \) and \( a \cdot b > 0 \), \( \forall a \in A \). Show that the existence of this separating hyperplane implies that there exists an arbitrage portfolio.
3. *(consumer optimality and state prices)*

Prove the following Theorem:

If there exists a solution to the consumer problem for an agent who prefers more to less, then there is no arbitrage. If $U$ is continuous and there is no arbitrage, then there exists a solution to the consumer problem.

Make whatever reasonable assumptions you like. For instance, it may be helpful to assume a particular class of preferences (i.e., additively separable across states of nature and state-independent). A rigorous answer is required: simply stating what is actually quite obvious is insufficient.

Note that, given the proofs we did in class and your proof of the above, we now have that the following are (pretty much) equivalent statements.

- There exist no arbitrage opportunities
- There exist positive state prices
- There exists some agent who prefers more to less and whose choice problem has a solution.