1. (arbitrage and state prices). Consider the following price, payoff pairs.

(a) \( q = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), \( D = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \)

(b) \( q = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \), \( D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \)

(c) \( q = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \), \( D = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \)

i. Which of these pairs are arbitrage-free?

ii. Compute, for each example, the complete set of state prices, \( \psi \), satisfying \( q = D\psi \).

iii. Use your answers to (i) and (ii) to verify that absence of arbitrage implies positive state prices.
2. (*binomial option pricing*). Consider the following single-period world. There are two equally likely states of the world, indexed by \( u \) and \( d \). There is a stock which trades at the beginning of the period for price \( S \). In addition to indexing the state of the world, the parameters \( u \) and \( d \) denote the state-contingent rate of return on the stock. Thus, the payoff to the stock-holder is \( Su \) in the \( u \) state of the world and \( Sd \) in the \( d \) state. There is also a riskless security (a discount bond) which pays one unit in either state of the world and costs \( B \) at the beginning of the period. Finally, there is an option which initially costs \( C \), has exercise price \( X \) and state-contingent payoff \((S' - X)^+\), where \( S' \) denotes the terminal value of the stock and \( x^+ \equiv \max(0, x) \). Denote \( R \) as one plus the riskless rate of interest.

(a) Derive necessary and sufficient conditions on the parameters \( u \), \( d \) and \( R \) such that arbitrage between the stock and bond markets is ruled out. Given that your conditions are violated, demonstrate how to construct an arbitrage portfolio.

(b) Solve for the option price, \( C \), in terms of \( X \), \( R \), \( S \), \( u \), and \( d \) in each of the following (equivalent) ways:

i. By constructing and using state prices.

ii. By constructing and using risk-neutral probabilities.

iii. By creating a *synthetic option* using the bond and stock and arguing that, in order to preclude arbitrage opportunities, the actual option must have the same price.

iv. By creating a riskless *hedge portfolio* using the option and the stock, and then arguing that, in order to preclude arbitrage opportunities, this portfolio must pay the same rate of return as the bond.

Note that the latter two ‘methods’ are how option pricing is initially taught in introductory textbooks.