Due: Thursday 3/3 at 5pm if you don’t go to the skating party, Friday 3/4 at 5pm if you do.

The rules are that, prior to turning your paper in, you are not to consult with anyone concerning this exam. You are not to consult course materials from years prior to this year.

This exam has two purposes: to teach you some more things and to evaluate you. The former is more important than the latter. In addition, the purpose of this exam is not to force you to spend 48 sleepless hours working like a dog. I may have overdone it in terms of quantity and/or difficulty: I’m not sure. If you are finding it difficult, that means that it probably is and that your classmates are also finding it difficult. Just do your best without knocking yourself out.

1. (non-diversifiable idiosyncratic risk and asset pricing)

Consider agent $i$’s first order condition expressed in terms of the excess return on a risky asset over a riskfree asset,

$$E_t (u'(c_{i,t+1})(R_{t+1} - R_f)) = 0.$$  

Assume that all agent’s have consumption processes with the same mean. By taking a second order approximation of marginal utility, and using it in this first-order condition, derive conditions under which the variance of the cross-sectional distribution of consumption (across agents) will tend to drive up the excess return on the risky asset. Comment on the economic intuition behind your result (HINT: at some point you will want to sum across agents).
2. (decentralized competitive equilibrium)

The environment is as follows.

- There are two dates, 0 and 1. Uncertainty at date 1 takes the form of two equally-likely states of nature: state 1 and state 2.
- Two agents, agent 1 and agent 2, have preferences
  \[ U_i(c) = \log c_{i0} + \beta_i E \log c_i \quad i, j = 1, 2 \]
  where \( c_{i0} \) is agent \( i \)'s consumption at date 0 and \( c_i \) is agent \( i \)'s consumption at date 1. \( \beta_i \) is the utility discount factor, where \( \beta_1 = 0.8 \) and \( \beta_2 = 0.9 \).
- Agents have non-tradeable labor income as follows:
  \[
  y_1 = \begin{bmatrix} y_{10} & y_{11} & y_{12} \end{bmatrix} = \begin{bmatrix} 10 & 12 & 8 \end{bmatrix}
  \\
  y_2 = \begin{bmatrix} y_{20} & y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} 10 & 7 & 13 \end{bmatrix}
  \]
  where \( y_{i0} \) is labor income for agent \( i \) at date 0 and \( y_{ij} \) is labor income for agent \( i \) at date 1 in state \( j \).
- Asset markets consist of a stock and a bond. The bond pays 1 unit of consumption at date 1 regardless of the realized state of nature. The stock pays 11 in state 1 and 9 in state 2. The bond is in zero net supply. There is one share of the stock outstanding. Agent 1 is endowed with 1/2 share and agent 2 with 1/2 share.

(a) Write down an agent’s choice problem and characterize the first-order conditions.

(b) Define a competitive equilibrium and show the equations which describe it.

(c) Solve for the equilibrium allocations and asset prices to the best of your ability. It is possible to derive numerical answers. However, algebraic expressions which capture the essence of what’s going on will get high marks.

(d) Describe the economic forces underlying the differences in the consumption allocations between agents 1 and 2.

(e) What is the risk premium in this economy?
3. (log-normal, consumption-based bond pricing)

Consider a stationary, infinite horizon representative agent economy with the following characteristics.

- Time is discrete and is indexed by \( t = 0, 1, 2, \ldots, \infty \)
- The agent’s preferences are,
  \[
  U(c) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),
  \]
  where \( u : \mathbb{R}_+ \rightarrow \mathbb{R} \) is strictly concave and everywhere differentiable, and \( \delta = 1/\beta - 1 \) is the agent’s utility rate of time preference.
- The agent is endowed with a strictly positive income process, \( \{e_t\}_{t=0}^{\infty} \).
- There are \( N \) assets, with ex-dividend prices \( s_i^t \), each of which is a claim to a divided process, \( \{d_i^t\}_{t=1}^{\infty} \). Denote the number of units of asset \( i \) held between periods \( t \) and \( t+1 \) as \( \theta_i^t, i = 1, 2, \ldots, N \).

(a) Characterize the agent’s dynamic programming problem. Use the first-order conditions from this problem to characterize the portfolio and consumption optimality conditions.

(b) Suppose that \( u(c) = (c^{1-\alpha} - 1)/(1-\alpha) \). Denote the aggregate endowment process for this economy as \( \{y_t\}_{t=0}^{\infty} \). Suppose that \( y \) obeys the following stationary stochastic process:

\[
\log(y_{t+1}/y_t) = \mu(1-\varphi) + \varphi \log(y_t/y_{t-1}) + \sigma \varepsilon_{t+1},
\]
where \( \varepsilon_t \sim N(0,1) \).
Consider two pure discount bonds. The first pays one unit of consumption one period from issuance and has an initial price, at some time \( t \), of \( b_1^t \). The second pays one unit of consumption two periods after being issued and has initial price \( b_2^t \). The yields-to-maturity on these bonds are defined as, respectively, \( r_1^t = -\log(b_1^t) \) and \( r_2^t = -\log(b_2^t)/2 \). Derive expressions for \( r_1^t \) and \( r_2^t \) in terms of the parameters of this model.
(c) Interpret $y_t$ as aggregate U.S. consumption. Here are some simple summary statistics for annual U.S. consumption growth and U.S. interest rates.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>St Dev</th>
<th>First-order Autocorrelation</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 month T-bill</td>
<td>6.08 %</td>
<td>3.17 %</td>
<td>0.98</td>
</tr>
<tr>
<td>24 month T-bill</td>
<td>6.27 %</td>
<td>3.12 %</td>
<td>0.97</td>
</tr>
<tr>
<td>Consumption Growth</td>
<td>1.80 %</td>
<td>3.60 %</td>
<td>-0.14</td>
</tr>
</tbody>
</table>

Using the expressions you derived in part (b), examine the relationship between the the average slope of the theoretical yield curve (the average difference between long and short rates) and the time series properties of the short rate and consumption growth in the model. Comment on which feature, or features, of the model are strongly at odds with the above data.

(d) Suppose that you are an econometrician who can only observe data on long and short term interest rates. Demonstrate how you would go about estimating a lower bound on the unconditional variance of any valid pricing kernel which is consistent with your dataset. Furthermore, given that the above model is true, derive an expression for the number which your estimated lower bound will converge to as your sample gets arbitrarily large (this number is the population Hansen-Jagannathan bound associated with these data).

Note: this part of the question does not require you to use information from the above table; algebraic expressions are all that are required.

4. (planning problem with incomplete markets)

Part (a) you have done on a problem set. Your job here is to do part (b).

- Two periods: today and tomorrow
- Two equally likely states-of-the-world are possible tomorrow: state 1 and state 2. Today will be referred to as state 0.
- Two agents, labeled agent 1 and agent 2, derive utility from the consumption of a single good. Denote $c \in \mathbb{R}^3_{++}$ as agent 1’s consumption and $d \in \mathbb{R}^3_{++}$ as agent 2’s consumption. Similarly, $c_i$ and $d_i$, $i = 0, 1, 2$, denote the state-specific consumption of agent 1 and 2,
respectively. Preferences are as follows.

\[
U_1(c) = \log(c_0) + \frac{\beta_1}{2} (\log(c_1) + \log(c_2))
\]

\[
U_2(d) = \log(d_0) + \frac{\beta_2}{2} (\log(d_1) + \log(d_2))
\]

Agent’s 1 and 2 have rates of time preference such that \(\beta_1 = 0.8\) and \(\beta_2 = 0.9\)

- The agents have endowments as follows.

\[
\tilde{c} = [\tilde{c}_0 \ \tilde{c}_1 \ \tilde{c}_2] = [1 \ 2 \ 1]
\]
\[
\tilde{d} = [\tilde{d}_0 \ \tilde{d}_1 \ \tilde{d}_2] = [1 \ 1 \ 2]
\]

where \(\tilde{c}\) denotes agent 1’s endowment and \(\tilde{d}\) denotes agent 2’s endowment.

(a) Assume that there are two securities marketed in this environment. Security 1 pays off one unit of the good if state one occurs and zero if state two occurs. Security 2 pays off one unit of the good if state 2 occurs and zero if state 1 occurs.

Construct a representative agent for this economy and use your construction to characterize the competitive equilibrium. That is, if we denote the aggregate endowment as \(e \in \mathbb{R}^3_+\), you are to construct a function, \(U_\lambda(e), \lambda \in \mathbb{R}_++\), such that no-trade is an equilibrium for this ‘aggregate economy,’ given the prices which solve the decentralized problem.

Your answer should use the representative agent construction to characterize the equilibrium allocation of consumption among the two agents. You should also clearly demonstrate the validity of the representative agent construction for this economy (i.e., show that no-trade is an equilibrium at the market clearing prices).

(b) Next, assume that the only security which is marketed is security number 1. Characterize a representative agent for this economy by using state-dependent weights in the utility function of the representative agent. Use this construction to characterize equilibrium prices and quantities in the same manner as you did in part (a).